Genus two partition functions of extremal conformal field theories

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ABSTRACT: Recently Witten conjectured the existence of a family of "extremal" conformal field theories (ECFTs) of central charge $c = 24k$, which are supposed to be dual to three-dimensional pure quantum gravity in AdS_3. Assuming their existence, we determine explicitly the genus two partition functions of $k = 2$ and $k = 3$ ECFTs, using modular invariance and the behavior of the partition function in degenerating limits of the Riemann surface. The result passes highly nontrivial tests and in particular provides a piece of evidence for the existence of the $k = 3$ ECFT. We also argue that the genus two partition function of ECFTs with $k \leq 10$ are uniquely fixed (if they exist).

Contents

1. Introduction

Recently Witten [1] argued that pure three-dimensional quantum gravity with a negative cosmological constant in AdS_3 should be dual to a CFT on the boundary of central charge $(c_L, c_R) = (24k, 24k)$, where k is a positive integer. This CFT factorizes into a holomorphic CFT and an anti-holomorphic CFT, whose lowest dimensional primary field has dimension $k + 1$. Such CFTs are called extremal (ECFT) [2]. A $k = 1$ ECFT was constructed by Frenkel, Lepowsky and Meurman [3] as a $\mathbf{Z}_2$ orbifold of free bosons on the Leech lattice, giving rise to the monster module. It is not yet known whether the $k > 1$ ECFTs exist, and it is clearly of interest to either construct them or to disprove their existence.

It was shown in [1] that the partition function for a $k = 2$ ECFT, if exists, can be constructed on any genus g hyperelliptic Riemann surface, using the $(2g + 2)$-point function of twist operators in the 2-fold symmetric product of the $k = 2$ ECFT. The partition function constructed in this way is consistent in the sense that, in the limit where the Riemann surface degenerates, the partition function reduces to lower genus correlation functions in suitable ways.

For example, the genus one partition function is related to the four-point function of the twist field $\mathcal{E}$. The latter can be determined from the $\mathcal{E}(z)\mathcal{E}(0)$ OPE, which essentially encodes the operator spectrum of the CFT. The genus two partition function, on the other hand, is related to six-point function of twist fields, and encodes information about the three point function of primaries. It is an $\mathrm{Sp}(4, \mathbf{Z})$ modular form of weight $2k$, and is in fact χ_{10}^{-k} times an entire holomorphic Siegel modular form of weight $12k$ [1]. Here χ_{10} stands for the weight 10 Igusa cusp form. The $k = 1$ genus two partition function has been computed in [4]. For $k = 1, 2, 3$, there are a basis of 3, 8 and 17 linearly independent

entire $\text{Sp}(4, \mathbf{Z})$ modular forms of weight 12, 24 and 36, respectively. One can determine the coefficients of these basis modular forms by considering the limits where the genus two Riemann surface degenerates. One limit ("pairwise degeneration") is when a handle of the Riemann surface is pinched, corresponding to a pair of the twist fields collide. Another limit ("separating degeneration") is when the Riemann surface degenerates into two genus one surfaces touching at a point (or conformally equivalently, connected by a thin tube).

In practice one can simplify things by considering a limit where all three pairs of twist fields degenerate, so that the six-point function can be replaced by the three point function of operators appearing in the singular terms of the $\mathcal{E}(z)\mathcal{E}(0)$ OPE. The latter is of the form

$$\mathcal{E}(z)\mathcal{E}(0) = \frac{1}{z^{3k}}(1 + \text{Virasoro descendants}) + \frac{1}{z^{k-2}} \sum_i \mathcal{O}^+_{k+1,i} \mathcal{O}^-_{k+1,i} + \cdots \quad (1.1)$$

where $\mathcal{O}^\pm_{k+1,i}$ are primaries of dimension $k+1$ in the two copies of the ECFT. Without using any information of $\mathcal{O}_{k+1,i}$, one can determine certain singular parts of the six-point function of $\mathcal{E}$. This turns out to be sufficient to fix (in fact, "over"-determining) the $k=1$ and $k=2$ genus two partition functions completely, while for $k=3$ one can fix all but 3 linear combinations of the 17 coefficients of the Siegel modular forms.

On the other hand, at the separating degeneration, the leading divergence of the genus two partition function factorizes as the product of the partition functions of the two genus one Riemann surfaces. For $k=1,2$, this is indeed the case, as expected from [1]. It also provides a highly nontrivial check for our expression for the $k=2$ genus two partition function. For $k=3$, the factorization at the separating degeneration fixes the remaining 3 coefficients of the modular forms, and hence the genus two partition function. Once again, it in fact "over"-determines the genus two partition function, hence the consistent factorization of our expression provides nontrivial evidences for the existence of the $k=3$ ECFT.

A slightly different approach, suggested in [1] as well, is to start by constructing the genus two partition function by sewing two genus one Riemann surfaces. Combining the knowledge of certain torus one-point functions and the $\text{Sp}(4, \mathbf{Z})$ modular invariance, we will show that it is in fact possible (at least in principle) to fix the genus two partition functions completely (and uniquely), for ECFTs with $k \leq 10$, assuming their existence. The explicit solutions, as well as consistency checks at all degenerations of the genus two Riemann surface, will be left to future work.

Section 2 describes some useful formulae for the OPE of twist fields and Siegel modular forms. In section 3, we shall examine the partition function of ECFTs with $k=1,2,3$. In section 4, we discuss the factorization at the separating degeneration to higher orders, and a general sewing construction of the genus two partition function.

2. Generalities

The 1-loop partition function of an extremal CFT M can be related to the 4-point function

of twistor operators in the symmetric product CFT $\text{Sym}^2(M)$ [5–8, 1],

$$Z = 2^{8k} \left(\prod_{1 \leq i < j \leq 4} e_{ij} \right)^k \langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}(e_4) \rangle. \tag{2.1}$$

where $\mathcal{E}$ is normalized such that $\langle \mathcal{E}(x)\mathcal{E}(0) \rangle = x^{-3k}$. The e_i's are related to the modulus τ of the torus as follows. If we set $e_4 = \infty$, $e_1 + e_2 + e_3 = 0$ by a conformal transformation, then the Jacobi theta functions of τ are related by

$$e_{12} = \theta_3(\tau)^4, \quad e_{32} = \theta_2(\tau)^4, \quad e_{13} = \theta_4(\tau)^4. \tag{2.2}$$

For general e_i's, we can write the j-function of τ as

$$j(\tau(e_1, e_2, e_3, e_4)) = 2^5 \frac{(\theta_2^8 + \theta_3^8 + \theta_4^8)^3}{\theta_2^8 \theta_3^8 \theta_4^8} = 2^5 \frac{(e_{12}^2 e_{34}^2 + e_{13}^2 e_{24}^2 + e_{14}^2 e_{23}^2)^3}{(\prod_{i<j} e_{ij})^2} \tag{2.3}$$

where $e_{ij} \equiv e_i - e_j$.

The OPE of the twist fields of the form

$$\mathcal{E}(x)\mathcal{E}(0) \sim \frac{1}{x^{3k}}(1 + \text{descendants}) + \frac{1}{x^{k-2}}\mathcal{O}_{2k+2}(0) + \cdots \tag{2.4}$$

where $\mathcal{O}_{2k+2}$ is a primary field of dimension $2k+2$ in the untwisted sector of $\text{Sym}^2(M)$. By examining the three-point function $\langle \mathcal{E}\mathcal{E}\mathcal{O} \rangle$ one can see that $\mathcal{O}_{2k+2}$ is in fact proportional to $\sum_i \mathcal{O}_{k+1,i}^+ \mathcal{O}_{k+1,i}^-$, where $\mathcal{O}_{k+1,i}^{\pm}$ are the complete set of dimension $k+1$ primaries in the two copies of the ECFTs.

To determine the Virasoro descendants appearing in (2.4), we shall closely follow the discussion of [1], but will work to higher orders. Inserting a pair of twist fields $\mathcal{E}(e)$, $\mathcal{E}(-e)$ in a correlation function amounts to compute the correlation function on the covering Riemann surface $y^2 = (x + e)(x - e)$. The Virasoro descendants appearing in the r.h.s. of (2.4) can be determined by requiring that the corresponding state is annihilated by the difference of the Virasoro generators on the two branches of the covering Riemann surface. Let $u = x + y, v = x - y$. The equation defining the double cover of the x-plane branched at $\pm e$ is then $uv = e^2$. The holomorphic vector fields

$$V_n = 2^{-n}u^{n+1}\partial_u = -2^{-n}e^{2n}v^{1-n}\partial_v \tag{2.5}$$

define the Virasoro generators

$$Q_n^+ = \oint_{S_+} V_n T = \oint_{S_+} 2^{-n}u^{n+1}\frac{dx}{du}dx T_{xx} \tag{2.6}$$

on the upper sheet, and

$$Q_n^- = \oint_{S_-} V_n T = \oint_{S_-} 2^{-n}e^{2n}u^{1-n}\frac{dx}{du}dx T_{xx} \tag{2.7}$$

on the lower sheet, up to a constant term due to the anomaly in transforming T from u to x coordinate. The operators $\hat{Q}_n = Q_n^+ - Q_n^-$ should annihilate the state appearing

in the $\mathcal{E}(e)\mathcal{E}(-e)$ OPE. The constant terms in the $\hat{Q}_n$'s can be determined by requiring $[\hat{Q}_n, \hat{Q}_m] = (n-m)\hat{Q}_{n+m}$. For our purpose, we will need the expressions for $\hat{Q}_{0,1,2,3,4}$ up to terms of order $\mathcal{O}(e^8)$. They are given explicitly by

$$
\begin{aligned}
\hat{Q}_0 &= \left(L_0^+ - \frac{e^2}{2}L_{-2}^+ - \frac{e^4}{8}L_{-4}^+ - \frac{e^6}{16}L_{-6}^+ - \frac{5e^8}{128}L_{-8}^+ \right) \\
&\quad - \left(L_0^- - \frac{e^2}{2}L_{-2}^- - \frac{e^4}{8}L_{-4}^- - \frac{e^6}{16}L_{-6}^- - \frac{5e^8}{128}L_{-8}^- \right) + \cdots \\
\hat{Q}_1 &= \left(L_1^+ - \frac{3e^2}{4}L_{-1}^+ - \frac{e^4}{16}L_{-3}^+ - \frac{e^6}{32}L_{-5}^+ - \frac{5e^8}{256}L_{-7}^+ \right) \\
&\quad - \left(\frac{e^2}{4}L_{-1}^- - \frac{e^4}{16}L_{-3}^- - \frac{e^6}{32}L_{-5}^- - \frac{5e^8}{256}L_{-7}^- \right) + \cdots \\
Q_2 &= \left(L_2^+ - e^2 L_0^+ - 3ke^2 + \frac{e^4}{16}L_{-2}^+ - \frac{e^8}{256}L_{-6}^+ \right) - \left(\frac{e^4}{16}L_{-2}^- - \frac{e^8}{256}L_{-6}^- \right) + \cdots \\
\hat{Q}_3 &= \left(L_3^+ - \frac{5e^2}{4}L_1^+ + \frac{e^4}{4}L_{-1}^+ + \frac{e^6}{64}L_{-3}^+ + \frac{e^8}{256}L_{-5}^+ \right) - \left(\frac{e^6}{64}L_{-3}^- + \frac{e^8}{256}L_{-5}^- \right) + \cdots \\
\hat{Q}_4 &= \left(L_4^+ - \frac{3e^2}{2}L_2^+ + \frac{e^4}{2}L_0^+ - \frac{3ke^4}{2} + \frac{e^8}{256}L_{-4}^+ \right) - \frac{e^8}{256}L_{-4}^- + \cdots
\end{aligned}
\tag{2.8}
$$

where $L^{\pm}$ are the Virasoro generators in the two copies of the CFT, i.e. on the two sheets. The state $|\Psi\rangle$ of the form $(1 + \text{descendants})|0\rangle$ and annihilated by all the $\hat{Q}_m$'s is

$$
\begin{aligned}
|\Psi\rangle = &\left\{ 1 + \frac{e^2}{4}L_{-2} + \frac{e^4}{32}\left[L_{-4} + L_{-2}^2 + \frac{1}{6k}L_{-2}^+ L_{-2}^- \right] \right. \\
&+ \frac{e^6}{128}\left[L_{-2}L_{-4} + \frac{1}{6k}L_{-2}^+ L_{-2}^- L_{-2} + \frac{1}{3}L_{-2}^3 + \frac{1}{24k}L_{-3}^+ L_{-3}^- \right] \\
&+ \frac{e^8}{512}\left[\frac{7}{6}L_{-8} + \frac{1}{2}(L_{-2}L_{-6} + L_{-6}L_{-2}) + \frac{1}{4}L_{-4}^2 \right. \\
&\qquad\qquad + \frac{1}{6}(L_{-2}^2 L_{-4} + L_{-2}L_{-4}L_{-2} + L_{-4}L_{-2}^2) \\
&\qquad\qquad + \frac{1}{12}L_{-2}^4 + \frac{1}{12k}L_{-4}L_{-2}^+ L_{-2}^- + \frac{1}{12k}L_{-2}^2 L_{-2}^+ L_{-2}^- + \frac{1}{24k}L_{-2}L_{-3}^+ L_{-3}^- \\
&+ \frac{1}{k(60k+11)}\left((k+\frac{1}{3})L_{-4}^+ L_{-4}^- + \frac{5}{12}(L_{-2}^+)^2(L_{-2}^-)^2 - \frac{1}{4}(L_{-4}^+(L_{-2}^-)^2 + L_{-4}^-(L_{-2}^+)^2) \right) \Big] \\
&\left. + \mathcal{O}(e^{10}) \right\}|0\rangle
\end{aligned}
\tag{2.9}
$$

The corresponding operator is $(e = x/2)$

$$
\begin{aligned}
\Psi_x = &1 + \frac{x^2}{16}T + \frac{x^4}{2^{10}}\partial^2 T + \frac{x^4}{2^9}T * T + \frac{x^4}{3 \cdot 2^{10}k}T^+ T^- + \frac{x^6}{2^{14}}T * \partial^2 T \\
&+ \frac{x^6}{3 \cdot 2^{14}k}(T^+ * T^+ T^- + T^- * T^- T^+) + \frac{x^6}{3 \cdot 2^{13}}T * (T * T) + \frac{x^6}{3 \cdot 2^{16}k}\partial T^+ \partial T^- \\
&+ \frac{x^8}{2^{17}}\left\{ \frac{7}{6 \cdot 6!}\partial^6 T + \frac{1}{48}(T * \partial^4 T + \partial^4 T * T) + \frac{1}{16}\partial^2 T * \partial^2 T + \frac{1}{12}T * (T * (T * T)) \right.
\end{aligned}
$$

$$+\frac{1}{12}(T*(T*\partial^2 T)+T*(\partial^2 T*T)+\partial^2 T*(T*T))+\frac{1}{24k}\partial^2 T*(T^+T^-)$$
$$+\frac{1}{12k}T*(T*(T^+T^-))+\frac{1}{24k}T*(\partial T^+\partial T^-)+\frac{1}{4k(60k+11)}\left[(k+\frac{1}{3})\partial^2 T^+\partial^2 T^-\right.$$
$$\left.\left.+\frac{5}{3}(T^+*T^+)(T^-*T^-)-\frac{1}{2}(\partial^2 T^+T^-*T^-+\partial^2 T^-T^+*T^+)\right]\right\}+\mathcal{O}(x^{10}) \quad (2.10)$$

where the notation $A*B$ stands for $\mathrm{Res}_{z\to 0}\left[A(z)B(0)/z\right]$. Now we can express the OPE of twist fields as

$$\mathcal{E}(x/2)\mathcal{E}(-x/2) \sim \frac{1}{x^{3k}}\Psi_x(0)+\frac{const}{x^{k-2}}\sum_i \mathcal{O}_i^+\mathcal{O}_i^-(0)+\cdots \quad (2.11)$$

Let us consider the six-point function $\langle\mathcal{E}(e_1)\cdots\mathcal{E}(e_6)\rangle$. It is related to the genus two partition function by

$$Z_{k,g=2}(\Omega) = A_k\left[\prod_{1\leq i<j\leq 6}(e_i-e_j)^k\right]\langle\mathcal{E}(e_1)\cdots\mathcal{E}(e_6)\rangle \quad (2.12)$$

where the genus two Riemann surface is represented as the hyperelliptic curve

$$y^2 = \prod_{i=1}^6(x-e_i), \quad (2.13)$$

and A_k is a constant. $Z_{k,g=2}$ is an $\mathrm{Sp}(4,\mathbf{Z})$ modular form of weight $2k$. The six-point function has singularities that goes like $(e_i-e_j)^{-3k}$. Multiplying it by $\prod_{i<j}(e_i-e_j)^{3k} = \chi_{10}^{3k/2}$, one obtains an entire holomorphic $\mathrm{Sp}(4,\mathbf{Z})$ Siegel modular form of weight $12k$. The ring of such modular forms is generated by the Eisenstein series ψ_4, ψ_6 (with slightly different normalization, as defined below) and the cusp forms χ_{10}, χ_{12}. Following [9], one defines the projective invariants

$$A = \sum_{15\text{ perms}} e_{12}^2 e_{34}^2 e_{56}^2,$$
$$B = \sum_{10\text{ perms}} e_{12}^2 e_{23}^2 e_{31}^2 e_{45}^2 e_{56}^2 e_{64}^2,$$
$$C = \sum_{60\text{ perms}} e_{12}^2 e_{23}^2 e_{31}^2 e_{45}^2 e_{56}^2 e_{64}^2 e_{14}^2 e_{25}^2 e_{36}^2,$$
$$D = \prod_{1\leq i<j\leq 6} e_{ij}^2, \quad (2.14)$$

There is a ring homomorphism mapping Siegel modular forms to projective invariants. We can write the projective invariants corresponding to generating modular forms as (by an abuse of notation, we shall not distinguish the two)

$$\psi_4 = B,$$
$$\psi_6 = \frac{1}{2}(AB-3C),$$
$$\chi_{10} = D,$$
$$\chi_{12} = AD. \quad (2.15)$$

An important property of the Siegel modular form is its factorization at the separating degeneration of the genus two Riemann surface, where the off-diagonal component τ_{12} of the period matrix goes to zero. We shall use the parameter ϵ defined in [4, 10], related by $2\pi i \tau_{12} = -\epsilon + \mathcal{O}(\epsilon^3)$. In the $\epsilon \to 0$ limit,

$$
\begin{aligned}
\psi_4 &= \frac{1}{4} E_4(\tau_1) E_4(\tau_2) + \mathcal{O}(\epsilon^2), \\
\psi_6 &= \frac{1}{16} E_6(\tau_1) E_6(\tau_2) + \mathcal{O}(\epsilon^2), \\
\chi_{10} &= const \cdot \epsilon^2 \Delta(\tau_1) \Delta(\tau_2) + \mathcal{O}(\epsilon^4), \\
\chi_{12} &= 96 \Delta(\tau_1) \Delta(\tau_2) + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{2.16}
$$

3. Explicit results

3.1 The $k = 1$ extremal CFT

As a warm up exercise we shall revisit the genus one and genus two partition functions of the $k = 1$ extremal CFT. The genus one partition function is

$$
Z_1(q) = J(q),
\tag{3.1}
$$

where $J(q) = j(q) - 744$. Identifying the four point function with (3.1), we can expand the part of $\langle \mathcal{E}(x/2)\mathcal{E}(-x/2)\mathcal{E}(y/2+z)\mathcal{E}(-y/2+z)\rangle$ that is singular in x, y in powers of z,

$$
\langle \mathcal{E}(x/2)\mathcal{E}(-x/2)\mathcal{E}(y/2+z)\mathcal{E}(-y/2+z)\rangle = x^{-3}y^{-3} + \frac{3}{32}x^{-1}y^{-1}z^{-4} + \frac{3}{64}(xy^{-1}+x^{-1}y)z^{-6} + \cdots
\tag{3.2}
$$

This expression can indeed be reproduced from (2.11) by explicitly evaluating the two point function $\langle \Psi_x(0)\Psi_y(z)\rangle$.

The $k = 1$ genus two partition function is a linear combination of

$$
\frac{\psi_4^3}{\chi_{10}}, \qquad \frac{\psi_6^2}{\chi_{10}}, \qquad \frac{\chi_{12}}{\chi_{10}}.
\tag{3.3}
$$

In the limit $e_{12}, e_{34}, e_{56} \to 0$, the singular terms in the six-point function of $\mathcal{E}(e_i)$ can be determined using the $\mathcal{E}\mathcal{E}$ OPE (2.11), together with the three point function of terms up to order x^2 in Ψ_x.

By matching with these, one can fix the unique choice of the modular form (up to overall normalization),

$$
Z_{k=1,g=2}(\Omega) = \frac{A_1}{\chi_{10}}\left(\frac{41}{4608}\psi_4^3 + \frac{31}{1152}\psi_6^2 - \frac{3813}{2048}\chi_{12}\right)
\tag{3.4}
$$

This is indeed the same expression as in [4] (note the different convention for the generating forms: in [4] F_{12} is not a cusp form; it is a more general linear combination of χ_{12}, ψ_4^3 and ψ_6^2). In the limit $\epsilon \to 0$, one can check that (3.4) indeed factorizes as

$$
Z_{k=1,g=2}(\Omega) \to \frac{const}{\epsilon^2} J(\tau_1) J(\tau_2).
\tag{3.5}
$$

Note that $\Delta = (E_4^3 - E_6^2)/1728$, and $J = (41E_4^3 + 31E_6^2)/(72\Delta)$.

We can extract information about the three-point functions of primaries from the genus two partition function (3.4). For example, by expanding the six-point function

$$\langle \mathcal{E}(\frac{x}{2})\mathcal{E}\left(-\frac{x}{2}\right)\mathcal{E}\left(\frac{y}{2}+u\right)\mathcal{E}\left(-\frac{y}{2}+u\right)\mathcal{E}\left(\frac{z}{2}+v\right)\mathcal{E}\left(-\frac{z}{2}+v\right)\rangle \tag{3.6}$$

corresponding to (3.4), up to order $\mathcal{O}(x^{-3}yz)$ and $\mathcal{O}(xyz)$ respectively, and subtracting the contribution from the Virasoro descendants in Ψ_x,[1] one obtains $\sum_{i,j}\langle \mathcal{O}_i\mathcal{O}_j\rangle^2$ and $\sum_{i,j,k}\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle^2$, where $\mathcal{O}_i$ are the 196883 dimension 2 primaries. Normalizing the $\mathcal{O}_i$'s such that $\langle \mathcal{O}_i(z)\mathcal{O}_j(0)\rangle = \delta_{ij}z^{-4}$, we find

$$\frac{1}{196883}\sum_{i,j,k=1}^{196883}\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\rangle^2 = \frac{13858}{3z_{12}^4 z_{13}^4 z_{23}^4} \tag{3.7}$$

3.2 The $k = 2$ extremal CFT

The $k = 2$ extremal CFT, if exists, has 1-loop partition function

$$Z_2(q) = J(q)^2 - 393767, \tag{3.8}$$

By comparing with the six-point function of the twist operator $\mathcal{E}$, in particular, the three-point function $\langle \Psi_x(0)\Psi_y(u)\Psi_z(v)\rangle$ up to order $\mathcal{O}(x^{-2}y^{-2}z^0)$, we can uniquely fix the genus two partition function,

$$Z_{k=2,g=2}(\Omega) = \frac{A_2}{\chi_{10}^2}\left(\frac{574489}{12230590464}\psi_4^6 + \frac{1125863}{1528823808}\psi_4^3\psi_6^2 + \frac{159769}{764411904}\psi_6^4\right. \tag{3.9}$$
$$-\frac{17809159}{905969664}\psi_4^3\chi_{12} - \frac{6550529}{226492416}\psi_6^2\chi_{12} + \frac{91785533041}{154618822656}\chi_{12}^2$$
$$\left.-\frac{393767}{1572864}\psi_4^2\psi_6\chi_{10} + \frac{229938936071}{9663676416}\psi_4\chi_{10}^2\right)$$

This partition function has the correct singular behavior as $e_{12}, e_{34}, e_{56} \to 0$. Furthermore, as $\epsilon \to 0$, (3.9) indeed factorizes as

$$Z_{k=2,g=2}(\Omega) \to \frac{const}{\epsilon^4}Z_2(\tau_1)Z_2(\tau_2). \tag{3.10}$$

This is a highly nontrivial consistency check of (3.9), which was determined without implementing (3.10).

Similarly to the $k = 1$ case, we can expand the six point function (3.6) corresponding to (3.9), up to order $\mathcal{O}(x^0y^0z^0)$, and extract information about the three-point function of the primaries of dimension 3. There are 42987519 such primaries, denoted by $\mathcal{O}_i$, whose two-point functions are normalized as before. We find

$$\frac{1}{42987519}\sum_{i,j,k=1}^{42987519}\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\mathcal{O}_k(z_3)\rangle^2 = \frac{104725}{4z_{12}^6 z_{13}^6 z_{23}^6} \tag{3.11}$$

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[1]The three-point functions of various Virasoro descendants are rather messy, and are computed using a Mathematica program. The program is available upon request.

As a piece of numerology, the fact that (3.11) is almost an integer multiple of $z_{12}^{-6} z_{13}^{-6} z_{23}^{-6}$ suggests that all the dimension 3 primaries $\mathcal{O}_i$ may be in one irreducible representation of some symmetry group (possibly containing the monster group as a subgroup).

3.3 The $k = 3$ extremal CFT

The $k = 3$ extremal CFT has 1-loop partition function

$$Z_3(q) = J(q)^3 - 590651 J(q) - 64481279. \tag{3.12}$$

The genus two partition function is $1/\chi_{10}^3$ times a weight 36 Siegel modular form. There are 17 independent Siegel modular forms of weight 36: ψ_4^9, $\psi_4^6 \psi_6^2$, $\cdots$, $\chi_{10}^3 \psi_6$. It turns out that by comparing with the six-point function of the twist operator $\mathcal{E}$ (3.6), up to the terms of order $\mathcal{O}(x^{-3} y^{-3} z^{-1})$, which does not require the knowledge of correlation functions of the dimension 4 primaries, we can determine all but 3 linear combinations of the 17 coefficients. The remaining 3 coefficients can be fixed by demanding factorization in the limit $\epsilon \to 0$,

$$Z_{k=3,g=2} \to \frac{const}{\epsilon^6} Z_3(\tau_1) Z_3(\tau_2). \tag{3.13}$$

This is not obviously possible, since the factorization a priori over-determines the remaining 3 coefficients. Remarkably, we do find a unique and consistent solution:

$$
\begin{aligned}
Z_{k=3,g=2}(\Omega) = \frac{A_3}{\chi_{10}^3} \Bigg[&\frac{307082041}{1352605460594688} \psi_4^9 + \frac{1025849351}{112717121716224} \psi_4^6 \psi_6^2 \\
&+ \frac{579427513}{28179280429056} \psi_4^3 \psi_6^4 + \frac{36867719}{21134460321792} \psi_6^6 \\
&- \frac{9519543271}{66795331387392} \psi_4^6 \chi_{12} - \frac{15531189821}{8349416423424} \psi_4^3 \psi_6^2 \chi_{12} \\
&- \frac{1511576479}{4174708211712} \psi_6^4 \chi_{12} + \frac{328564579342237}{17099604835172352} \psi_4^3 \chi_{12}^2 \\
&+ \frac{85316215289123}{4274901208793088} \psi_6^2 \chi_{12}^2 - \frac{11321414397534479}{60798594969501696} \chi_{12}^3 \\
&- \frac{150649445}{38654705664} \psi_4^5 \psi_6 \chi_{10} - \frac{76160539}{9663676416} \psi_4^2 \psi_6^3 \chi_{10} \\
&+ \frac{878731318367}{1855425871872} \psi_4^2 \psi_6 \chi_{10} \chi_{12} + \frac{492299265760247}{1068725302198272} \psi_4^4 \chi_{10}^2 \\
&+ \frac{256516494599113}{267181325549568} \psi_4 \psi_6^2 \chi_{10}^2 - \frac{36705982837911919}{1266637395197952} \psi_4 \chi_{10}^2 \chi_{12} \\
&- \frac{4272745361794189}{118747255799808} \psi_6 \chi_{10}^3 \Bigg].
\end{aligned}
\tag{3.14}
$$

This can be regarded as a piece of evidence for the existence (and perhaps uniqueness) of the $k = 3$ ECFT. It would be also straightforward to extract the sum of squares of the three-point functions of dimension 4 primaries in the $k = 3$ ECFT, as in the $k = 1, 2$ cases; although, we did not attempt this since the computation is rather time-consuming (even with our Mathematica program!).

4. Factorization and sewing

4.1 Next to leading order at the separating degeneration

It is also possible to compute the less singular terms in the expansion in ϵ near the separating degeneration. In practice, the expansion is easier to set up by working with the six point functions of twist fields. In the limit where three twist fields $\mathcal{E}$ are brought together and replaced by a generic operator in the twisted sector, the six point function factorizes into two four point functions, each corresponding to a torus partition function. A four point function with a generic operator in the twisted sector $O^1_{-n_1/2} O^2_{-n_2/2} \cdots |\mathcal{E}\rangle$ roughly corresponds to a torus one point function of $O^1_{-n_1} O^2_{-n_2} \cdots |0\rangle$. For an extremal CFT the second nonzero operator in the twisted sector after $\mathcal{E}$ is $L_{-1} \cdot \mathcal{E}(z) = -\partial \mathcal{E}(z)$.

The factorization limit can be set up, for example, as the $t \to 0$ limit of

$$\langle \mathcal{E}(te_1)\mathcal{E}(te_2)\mathcal{E}(te_3)\mathcal{E}(1/f_1)\mathcal{E}(1/f_2)\mathcal{E}(1/f_3)\rangle. \tag{4.1}$$

For convenience we will choose $e_1 + e_2 + e_3 = f_1 + f_2 + f_3 = 0$. The leading term in the six-point function, of order $\mathcal{O}(t^{-3k})$, will be

$$t^{-3k}\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)\mathcal{E}'(\infty)\rangle\langle \mathcal{E}(0)\mathcal{E}(1/f_1)\mathcal{E}(1/f_2)\mathcal{E}(1/f_3)\rangle = t^{-3k}\frac{Z_{g=1}(\tau_1)Z_{g=1}(\tau_2)}{(e_{12}e_{23}e_{13})^k(\tilde{f}_{12}\tilde{f}_{23}\tilde{f}_{13})^k} \tag{4.2}$$

where $\mathcal{E}'$ stands for the operator $\mathcal{E}$ in the $u = 1/z$ frame, $\tau_1 = \tau(e_1, e_2, e_3, \infty)$, $\tau_2 = \tau(f_1, f_2, f_3, \infty)$, as in (2.3), and $\tilde{f}_{ij} \equiv f_i^{-1} - f_j^{-1}$. The first subleading term in (4.1), of order $\mathcal{O}(t^{1-3k})$, will be

$$\frac{t^{1-3k}}{3k}\langle \mathcal{E}(e_1)\mathcal{E}(e_2)\mathcal{E}(e_3)(L_{-1}\cdot\mathcal{E})'(\infty)\rangle\langle L_{-1}\cdot\mathcal{E}(0)\mathcal{E}(1/f_1)\mathcal{E}(1/f_2)\mathcal{E}(1/f_3)\rangle \tag{4.3}$$

$$= \frac{t^{1-3k}}{3k}\left[\frac{\partial_x Z_{g=1}(\tau(e_i, 1/x))|_{x=0}}{(e_{12}e_{23}e_{13})^k}\right]\left[\frac{(f_1 f_2 f_3)^{3k}\,\partial_x Z_{g=1}(\tau(f_i, 1/x))|_{x=0}}{(f_{12}f_{23}f_{13})^k}\right]$$

$$= \frac{t^{1-3k}}{3k}\frac{\frac{1}{2\pi i}\partial_\tau Z_{g=1}(\tau_1)\frac{1}{2\pi i}\partial_\tau Z_{g=1}(\tau_2)}{(e_{12}e_{23}e_{13})^k(\tilde{f}_{12}\tilde{f}_{23}\tilde{f}_{13})^k},$$

where the factor $1/3k$ comes from the normalization of $L_{-1}|\mathcal{E}\rangle$, and we have used the identity

$$\partial_x \tau(e_i, 1/x)|_{x=0} = \frac{2^8 \cdot 27 e_1 e_2 e_3 (e_1^2 + e_2^2 + e_3^2 - e_1 e_2 - e_2 e_3 - e_3 e_1)^2}{(e_{12}e_{23}e_{13})^2\,\partial_\tau j(\tau_1)}$$

$$= -\frac{E_6(\tau_1)}{E_4(\tau_1)}\frac{j(\tau_1)}{\partial_\tau j(\tau_1)}$$

$$= \frac{1}{2\pi i}. \tag{4.4}$$

Note that $\frac{1}{2\pi i}\partial_\tau Z(\tau)$ is the torus one-point function of the stress-energy tensor. In the examples of $k = 1, 2, 3$, one can rewrite the genus two partition functions (3.4), (3.9), (3.14) in the form of the six-point function (4.1) using (2.14), (2.15), and expand in t. The result indeed matches (4.2), (4.3) precisely. Note that t is related to the parameter ϵ of [4, 10] by $t \sim \epsilon^4$.

4.2 Genus two partition function from sewing tori

Generally, the genus two partition function of a holomorphic CFT of central charge $c = 24k$ with small ϵ (as defined in [4, 10]) can be expanded as

$$Z_{g=2}(\tau_1, \tau_2, \epsilon) = \sum_i \epsilon^{2\Delta_i - 2k} \langle \mathcal{A}_i \rangle_{\tau_1} \langle \mathcal{A}_i \rangle_{\tau_2} \tag{4.5}$$

where $\mathcal{A}_i$ are an orthonormal basis of operators, with dimension Δ_i, $\langle \cdots \rangle_\tau$ stands for the one-point function on a torus of modulus τ, with $\langle 1 \rangle_\tau = Z_{g=1}(\tau)$. For an ECFT, all the operators with $\Delta \leq k$ are Virasoro descendants of 1, and their torus one-point functions can be derived using Ward identities. One can also constrain the torus one-point function of a general primary field $\mathcal{O}$ of dimension $\Delta(> 0)$. It can be written as

$$\langle \mathcal{O} \rangle_\tau = \mathrm{Tr} \mathcal{O}_0 q^{L_0 - k}, \tag{4.6}$$

where $\mathcal{O}_0 = \oint \frac{dz}{2\pi i} \mathcal{O}(z)$ is the zero mode of $\mathcal{O}$. Furthermore, $\langle \mathcal{O} \rangle_\tau$ is a modular form of weight Δ. The trace in (4.6) does not receive contribution from Virasoro descendants of the vacuum, and hence the leading term in the q-expansion of (4.6) is of order $q^{(k+1)-k} = q$. Therefore $\langle \mathcal{O} \rangle_\tau$ is a cusp form of weight Δ, and can be non-vanishing only for $\Delta \geq 12$. We will not attempt to further constrain $\langle \mathcal{O} \rangle_\tau$, which requires the knowledge of three-point functions of the primaries.

If $k \geq 11$, we can in principle determine the singular part as well as the $\mathcal{O}(1)$ part of $Z_{g=2}(\tau_1, \tau_2, \epsilon)$ in the $\epsilon \to 0$ limit. If $k \leq 10$, we know that the torus one-point functions of the primaries with $\Delta \leq 11$ vanish, hence knowing the one-point function of the Virasoro descendants of 1, up to dimension 11, we can in principle fix the terms in $Z_{g=2}(\tau_1, \tau_2, \epsilon)$ up to $\mathcal{O}(\epsilon^{2(11-k)})$.

On the other hand, by modular invariance we expect $Z_{g=2}$ to take the general form

$$Z_{g=2}(\Omega) = \sum_{m=0}^{\lfloor 6k/5 \rfloor} \chi_{10}^{-k+m} P_{12k-10m}(\psi_4, \psi_6, \chi_{12}) \tag{4.7}$$

where $P_{12k-10m}(\psi_4, \psi_6, \chi_{12})$ is a polynomial in $\psi_4, \psi_6, \chi_{12}$ of homogeneous weight $12k - 10m$. The leading terms in the expansion of $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ in the small ϵ limit are given in (2.16). If we know the terms of order $\mathcal{O}(\epsilon^{-2k+2m})$ in (4.5), the polynomials $P_{12k-10m}$ are fixed correspondingly.

For example, the leading singularity ϵ^{-2k} comes from the term $\chi_{10}^{-k} P_{12k}$,

$$\epsilon^{-2k} (\Delta(\tau_1) \Delta(\tau_2))^{-k} P_{12k} \left(\frac{1}{4} E_4(\tau_1) E_4(\tau_2), \frac{1}{16} E_6(\tau_1) E_6(\tau_2), 96 \Delta(\tau_1) \Delta(\tau_2) \right) \tag{4.8}$$

Writing

$$\frac{E_4(\tau_i)^3}{1728\Delta(\tau_i)} = x_i, \qquad \frac{E_6(\tau_i)^2}{1728\Delta(\tau_i)} = x_i - 1, \quad i = 1, 2, \tag{4.9}$$

(4.8) can be put in the form

$$\epsilon^{-2k} \tilde{P}(x_1 x_2, (x_1 - 1)(x_2 - 1)) \tag{4.10}$$

for some polynomial $\tilde{P}$. On the other hand, the leading term in (4.5) is of the form

$$\epsilon^{-2k}H(x_1)H(x_2), \tag{4.11}$$

for some polynomial $H(x)$, since $Z_{g=1}(\tau_i)$ is a polynomial in $J(\tau_i) = 1728x_i - 744$. There is a unique way of rewriting (4.11) in the form (4.10), which determines $P_{12k}(\psi_4, \psi_6, \chi_{12})$.

Similarly, comparison with the subleading terms in ϵ in (4.5) will in principle[2] determine $P_{12k-10}, P_{12k-20}, \cdots$. For $k \geq 11$, this will fix the P's up to P_{2k}. The remaining $P_{2k-10}, \cdots, P_{2k-10\lfloor\frac{k}{5}\rfloor}$ are not determined in this approach. For $k \leq 10$, since one can determine the terms in (4.5) up to $\mathcal{O}(\epsilon^{2(11-k)})$, all the polynomials $P_{12k-10m}$ are fixed. Therefore the genus two partition functions of the ECFTs with $k \leq 10$ are in principle uniquely fixed. To check the consistency of these partition functions (which is not a priori obvious), one should consider the limit where a handle pinches, say by comparing with the six-point function of twist fields as discussed in previous sections, or with two-point functions on the torus (the self-sewing of [10]). The consistency checks and explicit computations are left to future work.

Acknowledgments

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